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Name of Star.	Number of Observations.	J. — W. R. A.	Number of Observations.	J. — W. Decl.
α Aquilæ ...	10	—0°05	9	+0°3
β Aquilæ ...	4	+0°02	4	—0°2
α^1 Capricorni ...	—	—	2	—1°8
α^2 Capricorni ...	5	+0°01	5	—2°8
α Cygni ...	8	—0°01	6	—2°4
α Cephei... ..	11	—0°07	7	+0°9
β Cephei... ..	4	—0°11	10	+0°2
α Aquarii ...	9	—0°06	2	—1°0
α Piscis Australis	8	—0°02	8	—0°4
α Pegasi	4	—0°12	6	—0°3
α Ursæ Minoris...	8	—0°24	4	0°0
δ Ursæ Minoris...	4	+0°48	9	+1°8

Berlin, November 30, 1861.

On the Secular Acceleration of the Moon's Mean Motion.
By A. Cayley, Esq. (Abstract.)

The inclination and excentricity of the Moon's orbit, and *à fortiori* the variation of the position of the ecliptic, and the latitude of the Sun, are neglected; and the longitudes are measured from a fixed point in the ecliptic. The notation is in the present Abstract explained in so far only as it presents any peculiarity. I take

n , the actual mean motion of the Moon at a given epoch;

viz., it is assumed that the mean longitude at the time t is $\epsilon + nt + n_2 t^2 + \&c.$ where $\epsilon, n, n_2, \&c.$ are absolute constants; and, moreover,

a , the calculated mean distance of the Moon;

that is, $n^2 a^3$ is the sum of the masses of the Earth and Moon;
 a is, therefore, an absolute constant.

$$m, = \frac{n'}{n}, \text{ is an absolute constant.}$$

The Sun is considered as moving according to the law of elliptic motion in an orbit the excentricity whereof is $e' + \delta e'$, or $e' + f' t$, where e', f' , are absolute constants.

In the expression of the disturbing function the Sun's mass is taken to be $= n^{1/2} a'^3$, or, what is the same thing, $= m^2 n^2 a'^3$.

Then r , v , being the radius vector and longitude of the Moon, and taking the usual approximate expression of the disturbing function, the equations of motion are

$$\frac{d}{dt} \frac{dr}{dt} - r \left(\frac{dv}{dt} \right)^2 + \frac{n^2 a^3}{r^2} = m^2 n^2 a'^3 \frac{r}{r'^3} \left(\frac{1}{2} + \frac{3}{2} \cos 2v - 2v' \right),$$

$$\frac{d}{dt} \frac{r^2 dv}{dt} = m^2 n^2 a'^3 \frac{r^2}{r'^3} \left(-\frac{3}{2} \sin 2v - 2v' \right);$$

or writing

$$\frac{r}{a} = \varrho, \quad \frac{r'}{a'} = \varrho',$$

we have

$$\frac{d}{dt} \frac{\delta \varrho}{dt} - \varrho \left(\frac{dv}{dt} \right)^2 + \frac{n^2}{\varrho^3} = m^2 n^2 P,$$

$$\frac{d}{dt} \left(\varrho^2 \frac{dv}{dt} \right) = m^2 n^2 Q,$$

where, for shortness,

$$P = \frac{\varrho}{\varrho'^3} \left(\frac{1}{2} + \frac{3}{2} \cos 2v - 2v' \right),$$

$$Q = \frac{\varrho^2}{\varrho'^3} \left(-\frac{3}{2} \sin 2v - 2v' \right).$$

The notation is then changed by writing $\varrho' + \delta \varrho'$, $v' + \delta v'$, in the place of ϱ' , v' ; using henceforward ϱ' , v' , to denote the elliptic values of the solar co-ordinates, as calculated with the constant excentricity e' ; and, moreover, $\varrho + \delta \varrho$, $v + \delta v$, in the place of ϱ , v ; using henceforward ϱ , v , to denote the values of the lunar co-ordinates obtained from the equations of motion by writing therein ϱ' , v' , instead of the complete values $\varrho' + \delta \varrho'$, $v' + \delta v'$. Then putting

$$\delta P = \frac{dP}{d\varrho} \delta \varrho + \frac{dP}{dv} \delta v + \frac{dP}{d\varrho'} \delta \varrho' + \frac{dP}{dv'} \delta v',$$

with a like value for δQ , and observing that the equations of motion are satisfied when $\delta \varrho$, δv , $\delta \varrho'$, $\delta v'$, are neglected, we have

$$\frac{d}{dt} \frac{d\delta \varrho}{dt} - \delta \varrho \left(\frac{dv}{dt} \right)^2 - 2\varrho \frac{dv}{dt} \frac{d\delta v}{dt} - \frac{2n^2}{\varrho^3} \delta \varrho = m^2 n^2 \delta P,$$

$$\frac{d}{dt} \left(\varrho^2 \frac{d\delta v}{dt} + 2\varrho \delta \varrho \frac{dv}{dt} \right) = m^2 n^2 \delta Q;$$

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or, integrating the second equation, and by means of such integral changing the form of the first equation, we have

$$\frac{d^2 \delta \xi}{dt^2} + n^2 \delta \xi = \left\{ n^2 + \frac{2n^2}{\xi^3} - 3 \left(\frac{dv}{dt} \right)^2 \right\} \delta \xi + m^2 n^2 \left\{ \delta P + \frac{2}{\xi} \frac{dv}{dt} \left(C + \int dQ dt \right) \right\},$$

$$\frac{d \delta v}{dt} = - \frac{2}{\xi} \frac{dv}{dt} \delta \xi + \frac{m^2 n^2}{\xi^3} \left(C + \int \delta Q dt \right),$$

where C is to be determined by the condition that $\frac{d \delta v}{dt}$ may contain no constant term (for otherwise nt would not be the whole term varying directly as the time, in the longitude $v + \delta v$). The values of ξ' , v' , $\delta \xi'$, $\delta v'$, are given by the theory of elliptic motion, and those of ξ , v , by the ordinary lunar theory, in which the excentricity of the solar orbit is treated as a constant; the integration of the last-mentioned equations leads to the values of $\delta \xi$, δv , and then the radius vector and longitude of the moon are $a(\xi + \delta \xi)$, $v + \delta v$, respectively.

We, in fact, have

$$\begin{aligned} \frac{1}{\xi} &= 1 + \frac{1}{6} m^2 \\ &+ \frac{3}{2} m^2 e' \cos g' \\ &+ m^2 \quad ,, \quad 2\tau \\ &+ \frac{7}{2} m^2 e' \quad ,, \quad 2\tau - g' \\ &- \frac{1}{2} m^2 e' \quad ,, \quad 2\tau + g' \end{aligned}$$

$$\begin{aligned} v &= nt + \epsilon \\ &- 3 m e' \sin g' \\ &+ \frac{11}{8} m^2 \quad ,, \quad 2\tau \\ &+ \frac{77}{16} m^2 e' \quad ,, \quad 2\tau - g' \\ &- \frac{11}{8} m^2 e' \quad ,, \quad 2\tau + g' \end{aligned}$$

$$(g' = c' m n t + \text{const.} \quad 2\tau = (2 - 2m) n t + \text{const.})$$

$$\begin{aligned}
\frac{1}{e} &= 1 \\
&+ e' \cos g' \\
v' &= n' t + \epsilon \\
&+ 2 e' \sin g' \\
\delta \frac{1}{e} &= 1 \quad t \cos g' \\
&+ 2 e' \quad ,, \quad 2 g' \\
\delta v' &= 2 \quad t \sin g' \\
&+ \frac{5}{2} e' \quad ,, \quad 2 g'
\end{aligned}$$

where, instead of $\delta e' = f' t$, $\delta e'$ has been put $= t$, a simplification which is employed throughout the working, the factor f' being restored in the final results.

The values in the first instance obtained for δe , δv , are

$\delta e = f'$ into as follows, viz.:

$$\begin{aligned}
&\frac{3}{2} m^2 e' t &+ \underbrace{n^{-1} \times}_{3 m^3 \sin g'} \\
&+ \frac{3}{2} m^2 t \cos g' & \\
&+ 5 m^2 e' \quad ,, \quad 2 \tau &- \frac{203}{12} m^2 e' \quad ,, \quad 2 \tau \\
&- \frac{7}{2} m^2 \quad ,, \quad 2 \tau - g' &+ \frac{61}{24} m^2 \quad ,, \quad 2 \tau - g' \\
&+ \frac{1}{2} m^2 \quad ,, \quad 2 \tau + g' &- \frac{91}{14} m^2 \quad ,, \quad 2 \tau + g' \\
&+ \frac{9}{2} m^2 e' \quad ,, \quad 2 g' &- 18 m^3 e' \quad ,, \quad 2 g' \\
&- 17 m^2 e' \quad ,, \quad 2 \tau - 2 g' &+ \frac{85}{6} m^2 e' \quad ,, \quad 2 \tau - 2 g' \\
&0 \quad ,, \quad 2 \tau + 2 g' &- \frac{3}{4} m^2 e' \quad ,, \quad 2 \tau + 2 g'
\end{aligned}$$

and

$\delta v = f'$ into as follows, viz.:

$$\begin{aligned}
&- \frac{3}{2} m^2 n e' t^2 \\
&- 3 t \sin g' &+ \underbrace{n^{-1} \times}_{-3 + 6 m^2 \cos g'}
\end{aligned}$$

$$\begin{aligned}
& -\frac{55}{8} m^2 e' \sin 2\tau & -\frac{74}{3} m^2 e' \cos 2\tau \\
& +\frac{77}{16} m^2 & ,, 2\tau - g' & +\frac{215}{48} m^2 & ,, 2\tau - g' \\
& -\frac{11}{16} m^2 & ,, 2\tau + g' & -\frac{257}{48} m^2 & ,, 2\tau + g' \\
& -\frac{9}{2} m e' & ,, 2g' & \left(-\frac{9}{4} + 18 m^2\right) e' & ,, 2g' \\
& +\frac{187}{8} m^2 e' & ,, 2\tau - 2g' & +\frac{2239}{96} m^2 e' & ,, 2\tau - 2g' \\
& 0 & ,, 2\tau + 2g' & -\frac{33}{32} m^2 e' & ,, 2\tau + 2g'
\end{aligned}$$

where it is to be noticed that the terms $t \cos \arg.$ and $t \sin \arg.$ arise from the substitution of $e' + f' t$ for e' in the coefficients of the terms $\cos \arg.$ and $\sin \arg.$ in the ordinary values of ϵ and v . The new periodic terms $\sin \arg.$ in $\delta \epsilon$ and $\cos \arg.$ in δv agree as to the arguments $g', 2\tau, 2\tau - g', 2\tau + g'$, with those obtained by Prof. Adams; those for the other arguments $2g', 2\tau - 2g', 2\tau + 2g'$, do not contribute to the formation of the term in m^4 of the acceleration.

Starting from the foregoing expressions of $\delta \epsilon$ and δv , and proceeding to calculate the non-periodic parts as far as m^4 , the results are found to be

$$\delta \epsilon = \left(\frac{3}{2} m^2 - \frac{1973}{32} m^4 \right) e' f' t,$$

where the number $\frac{1973}{32}$ is made up

$$-\frac{1973}{32} = \frac{381}{8} - \frac{495}{32} - \frac{15}{4} - \frac{647}{32} - \frac{1455}{16} + \frac{675}{32};$$

and

$$\delta v = \left(-\frac{3}{2} m^2 + \frac{3771}{64} m^4 \right) e' f' t^2,$$

where the number $\frac{3771}{64}$ is obtained by means of the before-mentioned number $\frac{1993}{32}$; but replacing this number $\frac{1973}{32}$ by its component terms, the number $\frac{3771}{64}$ is made up

$$\frac{3771}{64} = -\frac{381}{8} + \frac{495}{32} + \frac{15}{4} + \frac{647}{32} + \frac{1455}{16} - \frac{675}{32} + \frac{275}{16} + \frac{45}{16} - \frac{1455}{64}.$$

It is hardly necessary to remark that, f''^2 being neglected, $2 e' f' t = (e' + f' t)^2 - e'^2$, and $\therefore e' f' t^2 = \int [(e' + f' t)^2 - e'^2] dt$, where $(e' + f' t)^2 - e'^2$ is what is commonly represented by $(e'^2 - E'^2)$. The value of m is not that made use of by Prof. Adams; in fact,—

$$m \text{ (Adams)} = m \left\{ 1 + \left(\frac{3}{2} m^2 - \frac{3771}{64} m^4 \right) 2 e' f' t \right\}$$

but in the terms multiplied by f' the two values of m may be considered as identical, hence the foregoing expression of δv , in the notation of Prof. Adams, is

$$\delta v = \left(-\frac{3}{2} m^2 + \frac{3771}{64} m^4 \right) \int (e'^2 - E'^2) n dt$$

the result obtained by him. The number $\frac{1973}{32}$, the coefficient of m^4 in the expression δe , does not occur explicitly in his memoir, but is, in fact, deducible from the value there given for the reciprocal of the radius vector. The present abstract gives a complete outline of the method of investigation, and it was thought better to omit altogether the details of the working rather than present them in an incomplete manner.

The Transit of Mercury, Nov. 11, 1861.

(Extract of a Letter from Father Secchi to the Astronomer Royal, dated Rome, Nov. 16, 1861.)

The passage of *Mercury* on the solar disk has been observed here under very good circumstances, but almost by a miracle, because the sky was all overcast by clouds. Some details will be found in the *Comptes Rendus*, and more yet in the *Astronomische Nachrichten*. But it will not displease the Astronomical Society to know the principal results; they are the following:—

	^h	^m	^s
Interior contact, Sid. T. Rome ...	13	35	6.29
Or M.T. Rome	22	9	9.45
Centre (estimated), Sid. T.	13	36	10.19
Exterior contact, Sid. T.	13	37	14.09

The air was exceedingly calm, and the rupture of the narrow ring was made with the utmost tranquillity, leaving two very sharp cusps. The exterior edge of the Sun appeared in the narrow thread of light, before the rupture, more faint